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Analytical and numerical contributions of some tenth-order graphs containing vacuum polarization insertions to the muon (g-2) in QED.

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Abstract.

The contributions to the $g-2$ of the muon from some tenth-order (five-loop) graphs containing one-loop and two-loop vacuum polarization insertions have been evaluated analytically in QED perturbation theory, expanding the results in the ratio of the electron to muon mass (m_e/m_μ). Some results contain also terms known only in numerical form. Our results agree with the renormalization group results already existing in the literature.

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In this work we have calculated in analytical form the contributions to the muon g-2 from the five-loop graphs obtained inserting one-loop and two-loop vacuum polarizations on a single photon line of the second-order and fourth-order vertex graphs; the considered graphs are shown in figs.1 and 2. Following the method used in previous calculations of the contributions of three-loop and four-loop graphs containing vacuum polarizations [1][2], we found it convenient to express the results expanding them in the ratio of the electron and muon masses (m_e/m_μ). Due to the length of the expressions of the coefficients of the high power terms, we will list them only up to (m_e/m_μ).

The analytical expressions of the contributions to the muon anomaly of the graphs shown in figs.1 and 2, accounting for the proper multiplicity factors, are ($r \equiv m_e/m_\mu$):

$$\begin{aligned} a_\mu^{(5)}[\text{fig.1(a)}] =& \frac{8}{81}\ln^4 r + \frac{200}{243}\ln^3 r + \left(\frac{8}{81}\pi^2 + \frac{634}{243}\right)\ln^2 r + \left(\frac{16}{27}\zeta(3) + \frac{100}{243}\pi^2 \right. \\ &\left. + \frac{8609}{2187}\right)\ln r + \frac{2}{135}\pi^4 + \frac{100}{81}\zeta(3) + \frac{317}{729}\pi^2 + \frac{64613}{26244} \\ &- r\left[\frac{18203}{374220}\pi^4\right] + O(r^2\ln^4 r)\ , \end{aligned} \tag{1}$$

$$\begin{aligned} a_\mu^{(5)}[\text{fig.1(b)}] =& \left(\frac{32}{81}\pi^2 - \frac{952}{243}\right)\ln^3 r + \left(\frac{8}{81}\pi^2 - \frac{244}{243}\right)\ln^2 r + \left(\frac{32}{135}\pi^4 - \frac{104}{81}\pi^2 \right. \\ &\left. - \frac{7627}{729}\right)\ln r + \frac{16}{27}\pi^2\zeta(3) + \frac{8}{405}\pi^4 - \frac{476}{81}\zeta(3) - \frac{2593}{2187}\pi^2 + \frac{64244}{6561} \\ &+ O(r^2\ln^2 r)\ , \end{aligned} \tag{2}$$

$$\begin{aligned} a_\mu^{(5)}[\text{fig.1(c)}] =& \left(\frac{64}{9}\zeta(3) - \frac{32}{405}\pi^2 - \frac{1886}{243}\right)\ln^2 r + \left(-\frac{16}{81}\pi^4 + \frac{16}{135}\zeta(3) + \frac{21532}{6075}\pi^2 \right. \\ &\left. - \frac{57899}{3645}\right)\ln r + \frac{64}{9}\zeta(5) + \frac{32}{27}\pi^2\zeta(3) - \frac{106}{6075}\pi^4 + \frac{10732}{2025}\zeta(3) \\ &- \frac{148921}{91125}\pi^2 - \frac{1090561}{109350} + O(r^2\ln r)\ , \end{aligned} \tag{3}$$

$$\begin{aligned} a_\mu^{(5)}[\text{fig.1(d)}] =& \left(\frac{16}{135}\pi^4 - \frac{256}{189}\zeta(3) - \frac{151849}{15309}\right)\ln r - \frac{16}{9}\pi^2\zeta(3) + \frac{124}{8505}\pi^4 \\ &+ \frac{92476}{6615}\zeta(3) + \frac{143}{81}\pi^2 - \frac{46796257}{3214890} + O(r^2)\ , \end{aligned} \tag{4}$$

$$
\begin{aligned}
a_\mu^{(5)}[\text{fig.1(e)}] =& -\frac{1}{3}\ln^3 r + \left(\frac{2}{3}\zeta(3) - \frac{35}{18}\right)\ln^2 r + \left(\frac{25}{9}\zeta(3) - \frac{1}{6}\pi^2 - \frac{413}{108}\right)\ln r \\
&+ \frac{1}{9}\pi^2\zeta(3) + \frac{263}{108}\zeta(3) - \frac{35}{108}\pi^2 - \frac{439}{162} \\
&+ r\left[-\frac{2516}{945}\pi^3\beta_2 - \frac{3401}{2268}\pi^5 - \frac{11}{54}\pi^4\ln 2 - \frac{1549}{360}\pi^2\zeta(3)\right. \\
&\quad \left. + \frac{12195383}{28576800}\pi^4 + \frac{87998}{8505}\pi^3 + \frac{970237}{43740}\pi^2\right] + O(r^2\ln^4 r)\ ,
\end{aligned} \tag{5}
$$

$$
\begin{aligned}
a_\mu^{(5)}[\text{fig.1(f)}] =& \left(-\frac{2}{3}\pi^2 + \frac{119}{18}\right)\ln^2 r + \left(\frac{4}{3}\pi^2\zeta(3) - \frac{119}{9}\zeta(3) + \frac{1}{6}\pi^2 - \frac{13}{8}\right)\ln r \\
&+ \frac{1}{9}\pi^2\zeta(3) - \frac{2}{15}\pi^4 - \frac{61}{54}\zeta(3) + \frac{161}{216}\pi^2 + \frac{3661}{648} + O(r^2\ln^2 r)\ ,
\end{aligned} \tag{6}
$$

$$
\begin{aligned}
a_\mu^{(5)}[\text{fig.1(g)}] =& \left(-4\zeta(3) + \frac{2}{45}\pi^2 + \frac{943}{216}\right)\ln r + 8\zeta^2(3) - \frac{4}{45}\pi^2\zeta(3) \\
&+ \frac{1}{18}\pi^4 - \frac{3833}{540}\zeta(3) - \frac{5483}{5400}\pi^2 + \frac{8581}{3240} + O(r^2\ln r)\ ,
\end{aligned} \tag{7}
$$

$$
\begin{aligned}
a_\mu^{(5)}[\text{fig.1(h)}] =& \left(-\frac{14}{405}\pi^4 + \frac{128}{9}a_4 - \frac{16}{27}\pi^2\ln^2 2 + \frac{16}{27}\ln^4 2 + \frac{26}{27}\zeta(3) - \frac{16}{27}\pi^2\ln 2\right. \\
&\left. - \frac{164}{243}\pi^2 + \frac{673}{81}\right)\ln^2 r + \left(-\frac{146}{9}\zeta(5) + \frac{256}{9}a_5 - \frac{76}{27}\pi^2\zeta(3) + \frac{32}{81}\pi^2\ln^3 2\right. \\
&+ \frac{196}{405}\pi^4\ln 2 - \frac{32}{135}\ln^5 2 - \frac{8}{27}\pi^2\ln^2 2 + \frac{16}{3}\pi^2\ln 2 - \frac{44}{405}\pi^4 \\
&\left. + \frac{64}{3}a_4 + \frac{8}{9}\ln^4 2 - \frac{1213}{81}\zeta(3) - \frac{4873}{1458}\pi^2 + \frac{33335}{1944}\right)\ln r + U_1 - \frac{463}{27}\zeta(5) \\
&+ \frac{64}{3}a_5 + \frac{29}{135}\pi^4\ln 2 - \frac{161}{162}\pi^2\zeta(3) + \frac{8}{81}\pi^2\ln^3 2 - \frac{8}{45}\ln^5 2 \\
&+ \frac{301}{7290}\pi^4 - \frac{1696}{81}a_4 - \frac{436}{243}\pi^2\ln^2 2 - \frac{212}{243}\ln^4 2 - \frac{3211}{324}\zeta(3) \\
&- \frac{952}{243}\pi^2\ln 2 + \frac{2635}{648}\pi^2 + \frac{31603}{1944} + O(r^2\ln r)\ ,
\end{aligned} \tag{8}
$$

$$
\begin{aligned}
a_\mu^{(5)}[\text{fig.1(i)}] =& \left(\frac{175}{9}\zeta(5) + \frac{140}{27}\pi^2\zeta(3) - \frac{10}{9}\pi^4\ln 2 + \frac{14}{6075}\pi^4 + \frac{3328}{135}a_4 - \frac{416}{405}\pi^2\ln^2 2\right. \\
&\left. + \frac{416}{405}\ln^4 2 + \frac{1561}{675}\zeta(3) - \frac{940}{243}\pi^2 + \frac{36653}{1800}\right)\ln r + U_2 + \frac{10681}{540}\zeta(5) \\
&+ \frac{3328}{135}a_5 - \frac{416}{2025}\ln^5 2 - \frac{263}{81}\pi^2\zeta(3) + \frac{416}{1215}\pi^2\ln^3 2 - \frac{3679}{12150}\pi^4\ln 2 \\
&+ \frac{202289}{583200}\pi^4 + \frac{2176}{2025}a_4 - \frac{272}{6075}\pi^2\ln^2 2 + \frac{272}{6075}\ln^4 2 - \frac{2933101}{60750}\zeta(3) \\
&+ \frac{2984}{243}\pi^2\ln 2 - \frac{56345}{8748}\pi^2 + \frac{13416707}{364500} + O(r^2)\ ,
\end{aligned} \tag{9}
$$

$$
\begin{aligned}
a_\mu^{(5)}[\text{fig.1(j)}] =& \frac{1}{8}\ln^2 r + \left(-\frac{1}{2}\zeta(3) + \frac{5}{12}\right)\ln r + U_3 - \frac{5}{6}\zeta(3) + \frac{1}{48}\pi^2 + \frac{409}{1152} \\
&+ r\left[\frac{110353}{272160}\pi^6 + \frac{9725}{756}\pi^5\ln 2 + \frac{4642}{567}\pi^4\ln^2 2 - \frac{389}{8}\pi^3\zeta(3) - \frac{3376}{63}\pi^3 Q_{12}\right. \\
&\quad + \frac{1688}{189}\pi^3\beta_2\ln 2 - \frac{6224}{63}\pi^2\beta_4 - \frac{3376}{189}\pi^2\beta_2^2 + \frac{3376}{189}\pi^2 a_4 + \frac{422}{567}\pi^2\ln^4 2 \\
&\quad - \frac{1461799}{127008}\pi^5 - \frac{21247}{756}\pi^4\ln 2 - \frac{30997}{4410}\pi^3\beta_2 + \frac{557663}{15120}\pi^2\zeta(3) \\
&\quad \left. + \frac{621830879}{33339600}\pi^4 + \frac{23060}{189}\pi^3\ln 2 + \frac{18448}{189}\pi^2\beta_2 - \frac{21563}{13230}\pi^3 - \frac{94319}{22680}\pi^2\right] \\
&+ O(r^2\ln^3 r)\ ,
\end{aligned} \tag{10}
$$

$$
\begin{aligned}
a_\mu^{(5)}[\text{fig.1(k)}] = & \left(\frac{7}{270}\pi^4 - \frac{32}{3}a_4 + \frac{4}{9}\pi^2\ln^2 2 - \frac{4}{9}\ln^4 2 - \frac{13}{18}\zeta(3) + \frac{4}{9}\pi^2\ln 2 + \frac{41}{81}\pi^2\right. \\
& \left. - \frac{673}{108}\right)\ln r - \frac{7}{135}\pi^4\zeta(3) + \frac{64}{3}a_4\zeta(3) - \frac{8}{9}\pi^2\zeta(3)\ln^2 2 + \frac{8}{9}\zeta(3)\ln^4 2 \\
& + \frac{13}{9}\zeta^2(3) - \frac{8}{9}\pi^2\zeta(3)\ln 2 + \frac{73}{12}\zeta(5) - \frac{32}{3}a_5 - \frac{49}{270}\pi^4\ln 2 + \frac{7}{162}\pi^2\zeta(3) \\
& - \frac{4}{27}\pi^2\ln^3 2 + \frac{4}{45}\ln^5 2 + \frac{97}{3240}\pi^4 - \frac{32}{9}a_4 - \frac{2}{27}\pi^2\ln^2 2 - \frac{4}{27}\ln^4 2 \\
& + \frac{1985}{108}\zeta(3) - \frac{59}{27}\pi^2\ln 2 + \frac{1351}{1296}\pi^2 - \frac{6625}{1728} + O(r^2\ln r)\ ,
\end{aligned} \tag{11}
$$

$$
\begin{aligned}
a_\mu^{(5)}[\text{fig.2(a)}] = & \left(\frac{8}{27}\pi^2\ln 2 - \frac{20}{81}\pi^2 - \frac{4}{9}\zeta(3) + \frac{31}{27}\right)\ln^3 r + \left(\frac{11}{162}\pi^4 - \frac{32}{9}a_4 - \frac{8}{27}\pi^2\ln^2 2\right. \\
& \left. - \frac{4}{27}\ln^4 2 - \frac{14}{3}\zeta(3) + \frac{20}{9}\pi^2\ln 2 - \frac{158}{81}\pi^2 + \frac{115}{18}\right)\ln^2 r \\
& + \left(\frac{143}{18}\zeta(5) - \frac{64}{9}a_5 + \frac{41}{405}\pi^4\ln 2 + \frac{2}{9}\pi^2\zeta(3) + \frac{16}{81}\pi^2\ln^3 2 + \frac{8}{135}\ln^5 2\right. \\
& + \frac{119}{810}\pi^4 - \frac{160}{9}a_4 - \frac{40}{27}\pi^2\ln^2 2 - \frac{20}{27}\ln^4 2 - \frac{133}{9}\zeta(3) + \frac{442}{81}\pi^2\ln 2 \\
& \left. - \frac{1133}{243}\pi^2 + \frac{8719}{648}\right)\ln r + U_4 + \frac{547}{36}\zeta(5) - \frac{160}{9}a_5 - \frac{1}{27}\pi^2\zeta(3) \\
& + \frac{40}{81}\pi^2\ln^3 2 + \frac{41}{162}\pi^4\ln 2 + \frac{4}{27}\ln^5 2 - \frac{485}{5832}\pi^4 - \frac{1768}{81}a_4 - \frac{442}{243}\pi^2\ln^2 2 \\
& - \frac{221}{243}\ln^4 2 - \frac{7199}{486}\zeta(3) + \frac{3503}{729}\pi^2\ln 2 - \frac{34541}{8748}\pi^2 + \frac{31531}{2916} \\
& + r\left[\frac{101}{3072}\pi^4\right] + O(r^2\ln^3 r)\ ,
\end{aligned} \tag{12}
$$

$$
\begin{aligned}
a_\mu^{(5)}[\text{fig.2(b)}] = &\left(\frac{2}{27}\pi^4 - \frac{35}{6}\zeta(3) - \frac{16}{9}\pi^2\ln 2 + \frac{62}{81}\pi^2 + \frac{227}{54}\right)\ln^2 r + \left(-\frac{40}{27}\pi^2\zeta(3)\right. \\
&+\frac{409}{1080}\pi^4 + \frac{104}{9}a_4 + \frac{35}{27}\pi^2\ln^2 2 + \frac{13}{27}\ln^4 2 + \frac{1475}{162}\zeta(3) - \frac{616}{81}\pi^2\ln 2 \\
&\left.-\frac{187}{729}\pi^2 + \frac{11891}{972}\right)\ln r + U_5 - \frac{1411}{144}\zeta(5) + \frac{104}{9}a_5 - \frac{26}{27}\pi^2\zeta(3) \\
&- \frac{35}{81}\pi^2\ln^3 2 - \frac{803}{3240}\pi^4\ln 2 - \frac{13}{135}\ln^5 2 - \frac{24679}{58320}\pi^4 + \frac{2284}{81}a_4 \\
&+ \frac{1277}{486}\pi^2\ln^2 2 + \frac{571}{486}\ln^4 2 - \frac{11381}{1944}\zeta(3) - \frac{1858}{243}\pi^2\ln 2 + \frac{58045}{8748}\pi^2 \\
&+ \frac{136247}{5832} + O(r^2\ln r)\ ,
\end{aligned}
\tag{13}
$$

$$
\begin{aligned}
a_\mu^{(5)}[\text{fig.2(c)}] = &\left(\frac{385}{18}\zeta(5) - \frac{11}{9}\pi^4\ln 2 + \frac{77}{27}\pi^2\zeta(3) + \frac{139}{432}\pi^4 - \frac{1459}{135}\zeta(3) + \frac{302}{405}\pi^2\right. \\
&\left.+\frac{3067}{3240}\right)\ln r + U_6 + \frac{929}{32}\zeta(5) - \frac{683}{216}\pi^2\zeta(3) - \frac{319}{432}\pi^4\ln 2 \\
&+ \frac{4}{27}\pi^2\ln^2 2 + \frac{102541}{388800}\pi^4 - \frac{32}{9}a_4 - \frac{4}{27}\ln^4 2 - \frac{128483}{12150}\zeta(3) - \frac{58}{27}\pi^2\ln 2 \\
&+ \frac{407693}{437400}\pi^2 + \frac{621001}{72900} + O(r^2)\ ,
\end{aligned}
\tag{14}
$$

$$
\begin{aligned}
a_\mu^{(5)}[\text{fig.2(d)}] = &\left(-\frac{2}{3}\pi^2\ln 2 + \zeta(3) + \frac{5}{9}\pi^2 - \frac{31}{12}\right)\ln^2 r + \left(-2\zeta^2(3) + \frac{4}{3}\pi^2\zeta(3)\ln 2\right. \\
&-\frac{10}{9}\pi^2\zeta(3) - \frac{11}{108}\pi^4 + \frac{16}{3}a_4 + \frac{4}{9}\pi^2\ln^2 2 + \frac{2}{9}\ln^4 2 + \frac{47}{4}\zeta(3) \\
&\left.-\frac{55}{18}\pi^2\ln 2 + \frac{97}{36}\pi^2 - \frac{1225}{144}\right)\ln r + \frac{11}{108}\pi^4\zeta(3) - \frac{16}{3}a_4\zeta(3) \\
&- \frac{4}{9}\pi^2\zeta(3)\ln^2 2 - \frac{2}{9}\zeta(3)\ln^4 2 - 7\zeta^2(3) + \frac{10}{3}\pi^2\zeta(3)\ln 2 - \frac{143}{24}\zeta(5) \\
&+ \frac{16}{3}a_5 - \frac{167}{54}\pi^2\zeta(3) - \frac{4}{27}\pi^2\ln^3 2 - \frac{41}{540}\pi^4\ln 2 - \frac{2}{45}\ln^5 2 - \frac{1153}{12960}\pi^4 \\
&+ \frac{110}{9}a_4 + \frac{55}{54}\pi^2\ln^2 2 + \frac{55}{108}\ln^4 2 + \frac{461}{24}\zeta(3) - \frac{367}{108}\pi^2\ln 2 \\
&+ \frac{1871}{648}\pi^2 - \frac{3497}{432} \\
&+ r\left[\frac{101}{144}\pi^3\beta_2 - \frac{101}{144}\pi^4\ln 2 + \frac{707}{288}\pi^2\zeta(3) + \frac{9035}{13824}\pi^4\right. \\
&\qquad\left.-\frac{821}{864}\pi^3 - \frac{5081}{1296}\pi^2\right] + O(r^2\ln^3 r)\ ,
\end{aligned}
\tag{15}
$$

$$
\begin{aligned}
a_\mu^{(5)}[\text{fig.2(f)}] = &\left(-\frac{1}{18}\pi^4 + \frac{35}{8}\zeta(3) + \frac{4}{3}\pi^2\ln 2 - \frac{31}{54}\pi^2 - \frac{227}{72}\right)\ln r + \frac{1}{9}\pi^4\zeta(3) \\
&- \frac{35}{4}\zeta^2(3) - \frac{8}{3}\pi^2\zeta(3)\ln 2 + \frac{46}{27}\pi^2\zeta(3) - \frac{1027}{8640}\pi^4 - \frac{13}{3}a_4 \\
&- \frac{35}{72}\pi^2\ln^2 2 - \frac{13}{72}\ln^4 2 + \frac{923}{864}\zeta(3) + \frac{62}{27}\pi^2\ln 2 + \frac{163}{486}\pi^2 - \frac{4243}{1296} \\
&+ O(r^2\ln r)\ .
\end{aligned} \tag{16}
$$

Here $\zeta(p)$ is the Riemann ζ-function of argument p, $\zeta(p) \equiv \sum_{n=1}^{\infty} \frac{1}{n^p}$, a_k and β_k are the constants defined as $a_k \equiv \sum_{n=1}^{\infty} \frac{1}{2^n n^k}$ and $\beta_k \equiv \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^k}$, whose first values are respectively $\zeta(2) = \pi^2/6$, $\zeta(3) = 1.202\ 056\ 903...$, $\zeta(4) = \pi^4/90$, $\zeta(5) = 1.036\ 927\ 755...$ $\zeta(6) = \pi^6/945$, $a_4 = 0.517\ 479\ 061...$, $a_5 = 0.508\ 400\ 579...$ and $\beta_1 = \pi/4$, $\beta_2 = 0.915\ 965\ 594...$, $\beta_3 = \pi^3/32$, $\beta_4 = 0.988\ 944\ 551...$.

Q_{12} is the transcendentality-three constant defined as

$$
Q_{12} \equiv \int_0^{\pi/2} \ln^2\left(2\cos\left(\theta/2\right)\right) d\theta = 0.550\ 279\ 883\ 952...\ . \tag{17}
$$

The constants β_2, β_4 and Q_{12} come from the values of the polylogarithmic functions

$$
\mathrm{Li}_2(i) = -\frac{1}{8}\zeta(2) + i\beta_2\ ,
$$

$$
\mathrm{Li}_4(i) = -\frac{7}{320}\zeta(4) + i\beta_4
$$

and

$$
S_{12}(i) = \int_0^i \frac{\ln^2(1-t)}{2t}dt = \frac{29}{64}\zeta(3) - \frac{1}{4}\pi\beta_2 + i\left(\frac{1}{32}\zeta(2)\pi - \frac{1}{2}Q_{12}\right)\ .
$$

Eq.(8)-(10) and eq.(12)-(14) contain also six different combinations U_i of several integrals with transcendentality six containing products of logarithms, dilogarithms and trilogarithms which we were unable to calculate in analytical form; an example of such integrals is $\int_0^1 \frac{dx}{x+1}\mathrm{Li}_3(x)\mathrm{Li}_2(-x)$. We have calculated them by using numerical methods

to high precision:

$$
\begin{aligned}
U_1 &= \quad 1.833\,055\,809\,327\ldots\ , \\
U_2 &= -\,8.520\,012\,033\,995\ldots\ , \\
U_3 &= \quad 0.722\,470\,399\,216\ldots\ , \\
U_4 &= \ 15.183\,055\,006\,269\ldots\ , \\
U_5 &= \ 18.086\,098\,879\,723\ldots\ , \\
U_6 &= \ 42.269\,128\,989\,300\ldots\ .
\end{aligned}
$$

We expect that these combinations of integrals can be expressed in analytical form using the transcendentality-six constants, $\zeta(6)$, $\zeta^2(3)$, $\ln^6 2$, a_6, $\zeta(5)\ln 2$, etc.. [1], in analogy with similar integrals with transcendentality five.

The calculation of the contribution of the graph of Fig.2(e) involves many groups of integrals with transcendentality six and seven, and we were unable to complete it in analytical form without the knowledge of the explicit analytical value of these integrals. The contribution of this graph will be given later in numerical form.

The expansions in (m_e/m_μ) eqs.(1)-(16) present some features already found in the expressions of the three-loop and four-loop contributions to the muon g-2 [1][2]:

1) The expression of the contributions of the graphs containing only electron loops (eq.(1), (5), (10), (12) and (15)) have a linear (m_e/m_μ) term [2]. The coefficients of these terms are constant, being numerically respectively -4.74, -18.00, -20.38, 3.20, -2.81. As we expected, these numerical values are larger than those found in four-loop calculations [2]. The (m_e/m_μ) expansions of the contributions of the other graphs containing also muon loops begin with the $(m_e/m_\mu)^2$ term.

2) The coefficients of the expansions in (m_e/m_μ) of the contributions of the graphs containing only electron loops contain high powers of $\ln(m_\mu/m_e)$. As an example, in table 1 we have listed the numerical values and the maximum power of $\ln(m_\mu/m_e)$ of

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[1] We found that the numerical value of U_5 coincides with that of $\frac{160}{9}\zeta(6)$ for at least the first 28 significant figures.

[2] At present only the analytical expression of the contribution of the three-loop light-light graphs it is known to contain a term proportional to $(m_e/m_\mu)\ \ln(m_\mu/m_e)$.

the coefficients of the powers of (m_e/m_μ) of eq.(5). We found that the coefficients of the even powers have the maximum power of $\ln(m_\mu/m_e)$ equal or greater than the maximum power of the zero order coefficient; the coefficient of $(m_e/m_\mu)^4$ contains the highest power of $\ln(m_\mu/m_e)$. The coefficients of the odd powers (greater than one) contain at most a $\ln(m_\mu/m_e)$ term. The same behaviour of the coefficients of the expansions was found even in all similar sixth- and eighth-order contributions [1][2]; this observation could be useful in the analytical calculations of the contributions of other high-order graphs containing only electron loops in order to estimate the magnitude of the uncalculated terms of the expansions in (m_e/m_μ).

3) At the five-loop level new transcendental constants, β_4 and Q_{12}, appear in the coefficient of the (m_e/m_μ) term of the expression of the contribution of the graph of Fig.1(j), containing a double insertion of a fourth-order vacuum polarization on a second order vertex graph; in analogous way, the odd power π^3 and the constant β_2 appeared at the three-loop level [1] and four-loop level [2] respectively. This fact remarks that the not leading terms of the contribution of the graph of Fig.1(j) cannot be worked out from the not leading terms of simpler graphs, whereas the leading ones can be easily deduced using the renormalization group equations [3].

We have compared our results with the renormalization-group results of ref.[3]. The leading terms of our eq.(1) and (5) agree respectively with eq.(35) and (32) of ref.[3] The logarithmic and constant terms of our eq.(10) agree with eq.(25) of ref.[3], except that our expression contains the unknown transcendentality-six constant U_3, whereas the expression of ref.[3] contains the analytical term $\zeta^2(3)/2$; these terms are in perfect numerical agreement, so that the analytical expression of U_3 can be inferred.

In table 2 we have listed the numerical values of eqs.(1)-(16) obtained using the experimental value [4] $(m_\mu/m_e) = 206.768262(30)$ and taking into account the (not shown) terms up to $(m_e/m_\mu)^4$. The value of the contribution of the graph of Fig.2(e) has been worked out evaluating numerically the one-dimensional integral obtained using the dispersive representation for the vacuum polarization [2].

As expected, the graphs containing muon loops give contributions substantially smaller than those containing only electron loops. The sum of the numerical values of

the three graphs of fig.1 calculated using the renormalization group technique in ref.[3] has a rather large value, about 52. On the contrary the sum of all contributions of fig.1 and 2 shows a strong cancellation, the numerical value being

$$a_\mu^{(5)}[\text{fig.}1+2] = -1.261\,574(2)\ . \tag{18}$$

This fact is due to the cancellation between the positive contributions of the graphs of fig.1 containing vacuum polarization insertion on the second-order vertex graph and the negative contributions of the graphs of fig.2 containing vacuum polarization insertions on fourth-order vertex graphs. We found a similar behaviour at the four-loop level, even if in that case the cancellation was less marked [2]. Finally, eq.(18) turns out to be much smaller than the estimate of the total tenth-order contribution [5]

$$a_\mu^{(5)} = 570(140)\ . \tag{19}$$

All the algebraic manipulations were carried out through the symbolic manipulation program ASHMEDAI [6].

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Figure captions

Fig.1: Tenth-order vertex graph obtained with insertions of second- and fourth-order vacuum polarization subdiagrams on the second-order vertex graph.

Fig.2: Examples of tenth-order vertex graph obtained with insertions of second- and fourth-order vacuum polarization subdiagrams on the fourth-order vertex graphs.

TABLE 1: The numerical values and the maximum power of $\ln(m_\mu/m_e)$ of the coefficients of the expansion in (m_e/m_μ) of the contribution of the graph of Fig.1(e).

n	Coefficient of $(m_e/m_\mu)^n$	Maximum power of $\ln(m_\mu/m_e)$
0	27.71	3
1	-18.00	0
2	2493.30	4
3	154.49	0
4	-34783.30	5
5	-3.91	0
6	452.06	4
7	11.93	0
8	-3859.59	4
9	3.15	0
10	-1346.48	4

TABLE 2: The numerical values of the contributions to $a_\mu^{(5)}$ of the graphs of figs.1 and 2.

Figure	Contribution to $a_\mu^{(5)}$
1(a)	20.142 811(3)
1(b)	2.203 327 2(2)
1(c)	0.206 959 08(1)
1(d)	0.013 875 908 8(3)
1(e)	27.690 059 (3)
1(f)	1.166 152 15(6)
1(g)	0.031 814 813 1(6)
1(h)	1.614 350 1(1)
1(i)	0.164 714 747(4)
1(j)	4.742 149 1(2)
1(k)	0.399 245 484(8)
2(a)	− 28.429 744 (2)
2(b)	− 6.792 291 9(3)
2(c)	− 0.952 823 49(2)
2(d)	− 19.042 323 (1)
2(e)	− 3.131 386 2(1)
2(f)	− 1.288 464 12(2)
$1+2$	− 1.261 574(2)